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Laminar free convection flow at a stagnation point of attachment on an isothermal surface

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SUMMARY

Results are presented of a numerical study of the three-dimensional free convectional flow near a stagnation point of attachment on an isothermal surface when the Prandtl number is 0.72. The boundary layer flows that result are of both nodal and saddle point type. The results of the numerical integrations are presented graphically for various values of $c=b/a$, where b and a are the principal curvatures of the surface at the stagnation point. Two singularities are found to exist.

1. Introduction

When a fluid moves over a finite body there are points at which the fluid attaches itself to the body and other points where the fluid leaves the body. The method used to describe the nature of these particular points is by way of the behaviour of the skin-friction lines at the surface of the body [1].

Lines on the body surface whose tangent at a point coincides in direction with the skin-friction vector at that point are called the skin-friction lines. The vortex lines on the body surface are everywhere orthogonal to the skin-friction lines except at points of attachment or separation at which both skin-friction and vorticity vanish: the latter positions are singular points of both differential equations defining the skin-friction and vortex lines, and are called stagnation points. By choosing a co-ordinate system $Oxyz$ with origin at the singular point and Oz normal to the surface, the classification of such a point depends on the sign of the Jacobian, $J=\partial(\varepsilon_x, \varepsilon_y)/\partial(x, y)$ where $\mu\varepsilon=\mu(\varepsilon_x, \varepsilon_y)$ is the skin-friction vector in the plane Oxy. If $J>0$ the point is a nodal point and if $J<0$ it is a saddle point.

Further, if the normal velocity near to the origin is towards $z=0$ the stagnation point is one of attachment, while if the normal velocity is away from the surface it is a point of separation. It can be shown that, provided the velocity field is solenoidal, then depending on whether the two-dimensional divergence of ε is >0 or <0 the singular point is a stagnation point of attachment or separation respectively.

The forced convectional flow at nodal and saddle points of attachment has been discussed by Howarth [2] and Davey [3] respectively. The physical situation at which it is assumed these solutions are valid is indicated in figure 1, where N and S indicate geometrical nodal and saddle points respectively. The arrow shows the flow direction in order to have points of attachment. We note that the flow in the vicinity of these attachment points is described by a sixth-order system of ordinary differential equations which involves a parameter c: $c>0$ corresponds to nodal points of attachment and $c<0$ to saddle points of attachment.

The saddle point flows discussed by Davey are "terminal" solutions and the basic assumption that such flows are locally determinate have been verified for certain values of $c<0$ by Banks [4] and also by Cooke and Robins [5]. It should be noted, however, that for $c<-0.43$ Davey found that one of the flow velocities ceased to be unidirectional and suggested that such solutions were physically unrealistic. This interpretation is consistent with the later findings in [4] where it was found from a forward-integration calculation that for $0>c>-0.43$ Davey's stagnation point saddle flows were recovered, but that for $c<-0.43$ the skin-friction vanished

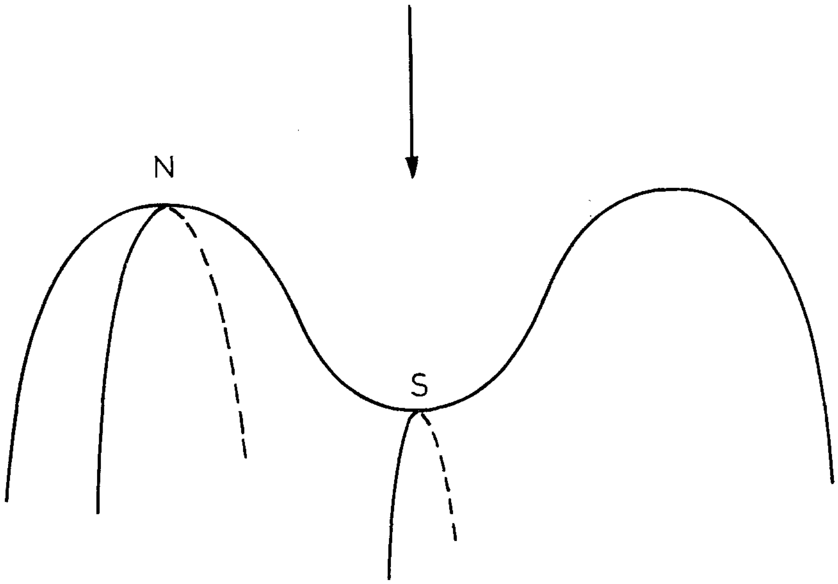


Figure 1. Typical nodal (N) and saddle (S) points. The arrow indicates the flow direction for forced flow and the direction of gravity for free convection flow.

and a singularity occurred before the boundary layer reached S.

The free convectional flow at a stagnation point of attachment on an isothermal surface has been examined by Poots [6]; the governing equations for this situation comprise an eighth-order system, which involves the Prandtl number, σ, in addition to the parameter c that describes the local geometry of the surface. Poots provided numerical solutions for $\sigma=0.72$ and various positive values of c which correspond to nodal points of attachment.

The purpose of the present paper is to extend the calculations of Poots to negative values of c corresponding to saddle points of attachment. The Prandtl number is again chosen to be 0.72.

2. The boundary layer equations

Poots [6] has shown that, by choosing a locally orthogonal set of co-ordinates $Oxyz$ at the isothermal body surface in such a way that the origin coincides with the stagnation point and the parametric curves $x=$constant, and $y=$constant on the surface coincide with the lines of curvature, the boundary layer equations can be written

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=g\beta(T-T_\infty)ax+\nu\frac{\partial^2 u}{\partial z^2}, \tag{1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=g\beta(T-T_\infty)by+\nu\frac{\partial^2 v}{\partial z^2}, \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=k\frac{\partial^2 T}{\partial z^2}, \tag{4}$$

where (u, v, w) are the velocities in the directions (x, y, z) increasing respectively, T is the temperature with T_∞ the temperature at infinity, g is the acceleration due to gravity, β the coefficient of cubical expansion, k is the thermal diffusivity and ν the kinematic viscosity. The two remaining parameters a and b are the curvatures of the body surface measured in the planes $y=0$ and $x=0$ respectively: because of the choice of axes a and b are the principal curvatures at 0.

The parameters a and b were taken to be non-negative by Poots so that the body is of nodal type at 0, and the solutions of the resulting equations found in [6] correspond to stagnation points which are nodal points of attachment. However, if one of a and b is negative, the body is of saddle type and it is to be assumed that certain solutions of the equations will correspond to saddle points of attachment. Clearly if both a and b are negative the body is again of nodal type but the flow is one of separation and is not considered here.

Without loss of generality, we therefore take b to be negative and, in terms of the Grashof number $G=\beta g(T_0-T_\infty)/a^3\nu^2$, where T_0 is the constant wall temperature, look for a solution by writing

$$u=\nu a^2 xG^{\frac{1}{2}}f'(Z)\,,\quad v=\nu a^2 yG^{\frac{1}{2}}g'(Z)\,,\quad w=-\nu aG^{\frac{1}{4}}(f+g)\,, \qquad (5)$$
$$T=T_\infty+(T_0-T_\infty)h(Z)\,,$$

where $Z=G^{\frac{1}{4}}az$. The continuity equation (3) is automatically satisfied by this choice and equations (1), (2) and (4) yield

$$f'''+(f+g)f''-f'^2+h=0\,, \qquad (6)$$
$$g'''+(f+g)g''-g'^2+ch=0\,, \qquad (7)$$
$$h''+\sigma(f+g)h'=0\,, \qquad (8)$$

where $c=b/a$, $\sigma=\nu/k$ is the Prandtl number and dashes imply differentiation with respect to Z. The non-dimensionalisation differs slightly from that of Poots to allow c to take positive and negative values. The problem is completely posed by adding the boundary conditions

$$f(0)=f'(0)=g(0)=g'(0)=0\,,\ h(0)=1\,, \qquad (9)$$
$$f'(Z)\to 0\,,\quad g'(Z)\to 0\,,\quad h(Z)\to 0 \text{ as } Z\to\infty\,.$$

As Poots points out, with $c=0$ we recover the well-known two-dimensional problem by assuming $g(Z)\equiv 0$, while with $c=1$ and assuming $f(Z)\equiv g(Z)$ we recover the axi-symmetric problem. Since Poots assumed that, in the present notation, both a and b were positive he was able to restrict attention to cases when $0\leqq c\leqq 1$ by a possible change of axes. This is equivalent to noting that equations (6), (7) and (8) imply that

$$f(c,Z)=c^{\frac{1}{4}}g(c^{-1},c^{\frac{1}{4}}Z)\,,\quad g(c,Z)=c^{\frac{1}{4}}f(c^{-1},c^{\frac{1}{4}}Z)\,,$$
$$h(c,Z)=h(c^{-1},c^{\frac{1}{4}}Z)\,. \qquad (10)$$

These relations give no additional solutions when applied to those provided by Poots, although in this study dual solutions in the range $0\leqq c<\infty$ are shown to arise and relations (10) will be used to "reflect" the solutions from the range $1<c<\infty$ into the range $0<c<1$ for convenience. It may be convenient at this stage to note that Poots gave numerical solutions to equations (6), (7) and (8) subject to (9) for values $c=0, \frac{1}{16}, \frac{1}{4}, \frac{9}{16}, 1$ and $\sigma=0.72$.

In view of the boundary conditions in (9) as $Z\to\infty$, it can be shown that for $\sigma\neq 1$ and large Z

$$f\sim\delta_1-A\lambda\,\mathrm{e}^{-\sigma\theta Z}+B\,\mathrm{e}^{-\theta Z}\,,$$
$$g\sim\delta_2-cA\lambda\,\mathrm{e}^{-\sigma\theta Z}+C\,\mathrm{e}^{-\theta Z}\,, \qquad (11)$$
$$h\sim A\,\mathrm{e}^{-\sigma\theta Z}\,,$$

where $\theta=\delta_1+\delta_2$ and $\lambda^{-1}=\sigma^2\theta^3(1-\sigma)$. For $\sigma=1$ analogous expressions can be obtained. Clearly we shall require $\theta>0$ in order that the solutions satisfy the boundary conditions and this implies the normal flow at the edge of the boundary layer is *towards* $Z=0$. Also they will be flows of *attachment* in the sense outlined in section 1 provided $f''(0)+g''(0)>0$ and we anticipate what is to follow by noting that all the solutions found in this study satisfy this condition.

A point of interest arises at this stage on comparison with the forced convection problem: it was found in [3] that for $c<0$ the solutions were clearly not unique and a device was adopted for selecting what was assumed to be the physical solution. Equations (11) on the other hand imply complete freedom from the latter sort of non-uniqueness. It is also clear from the leading terms of the asymptotic forms in (11) that for $0<\sigma<1$ the velocities u, v take opposite signs for large Z with $c<0$, while for $c>0$ they have the same sign. These properties are observed in all the numerical solutions found in this investigation.

In the forced convection problem the boundary layer is driven by the outer flow and it is found that the characteristics of this flow provides physical arguments for expecting a lower negative bound to the values of c below which no solutions exist. However, in the free convection situation there is no outer flow to this order, the driving mechanism varying across the boundary layer in a manner which is known only when the solution is known. Nevertheless one may, on physical grounds, expect a limiting value of c, say c^*, below which no solutions exist. We may note other evidence for the existence of c^*: if we assume $c<0$ then with $\gamma=-c$ the analogue of (10) is

$$f(\gamma, Z)=\gamma^{\frac{1}{4}} F(\gamma, \gamma^{\frac{1}{4}} Z), \quad g(\gamma, Z)=\gamma^{\frac{1}{4}} G(\gamma, \gamma^{\frac{1}{4}} Z),$$
$$h(\gamma, Z)=H(\gamma, \gamma^{\frac{1}{4}} Z),$$

and equations (6), (7) and (8) become

$$F'''+(F+G)F''-F'^2+\gamma^{-1}H=0 \;,$$
$$G'''+(F+G)G''-G'^2-H=0 \;,$$
$$H''+\sigma(F+G)H'=0,$$

where dashes now imply differentiation with respect to $\zeta=\gamma^{\frac{1}{4}} Z$, and the boundary conditions are as in (9). However for γ large, we may expect $F=O(1/\gamma)$ and the equations for G and H become

$$G'''+GG''-G'^2-H=0,$$
$$H''+\sigma GH'=0,$$

on neglecting F. These equations govern the steady flow at a point of separation and it appears to be generally accepted that they do *not* possess a solution.

In view of this it was decided to look for numerical solutions of equations (6), (7) and (8) subject to (9) for $c<0$, by continuously decreasing from $c=0$ and hence to find the limit c^* below which no solutions exist.

The method used in the numerical integrations was the "shooting" method, whereby for $\sigma=0.72$ and fixed c trial values of $f''(0)$, $g''(0)$, $h'(0)$ are perturbed until the boundary conditions for large Z are satisfied. The outer boundary condition was imposed at $Z=20$ for most values of c. A computer library procedure was used for the integrations which had an automatic change of step-length facility thus providing a check on the accuracy.

3. Numerical results

Because Poots had given solutions to equations (6), (7) and (8) subject to (9) with $\sigma=0.72$ for certain values of $c\geqq 0$, the method adopted was to continue the tabulation of $f(Z)$, $g(Z)$, $h(Z)$ to negative values of c for the same Prandtl number. Also, in view of the dual solutions found by Schofield and Davey [7] in forced convectional flow, the possibility of a dual solution existing at $c=0$ was at the same time investigated. This was found and reported upon elsewhere [8] and its existence was used to approach further solutions for $c<0$ and $c>0$ along a different branch of the solution curve.

The results of the numerical integrations are presented in graphical form. Figures 2 and 3 show respectively the variation of $f''(0)$, $g''(0)$, $h'(0)$ and $f(\infty)$, $g(\infty)$ etc. with c. We have, following Schofield and Davey, denoted the dual solutions by the subscript d for $c\leqq 1$. As expected on physical grounds we find $h(Z)\geqq 0$ for all flows. We note that the limit value $c^*=-0.1559\ldots$ and its defining characteristic would appear to be the singularity that is suggested by the numerical results. Thus, as in the well-known Falkner–Skan flows, in the neighbourhood

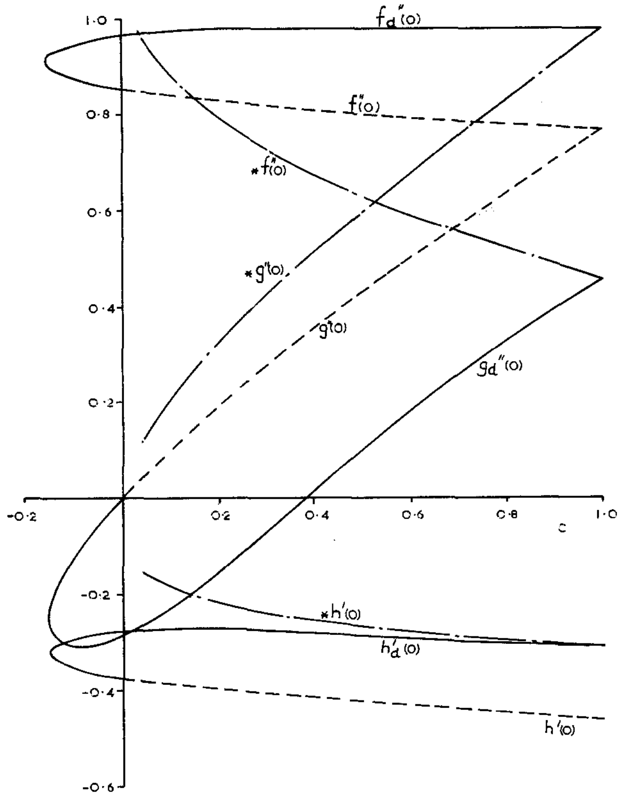


Figure 2. Values of $f''(0)$, $g''(0)$, $h'(0)$ for dual and non-dual solutions with $\sigma = 0.72$ and $c^* \leqq c \leqq 1$. The broken lines show results found by Poots [6]. The dotted lines show results of dual solutions corresponding to $c > 1$ (see text for further explanation).

of c^* the solution varies with c like $(c - c^*)^{\frac{1}{2}}$, and in particular

$$\begin{aligned} f''(0) &= 0.91165 - 0.14(c - c^*)^{\frac{1}{2}} , \\ g''(0) &= -0.25533 + 0.47(c - c^*)^{\frac{1}{2}} , \\ h'(0) &= -0.32142 - 0.13(c - c^*)^{\frac{1}{2}} , \end{aligned} \tag{12}$$

where $c^* = -0.1559$. No special calculations were made to find very detailed behaviour near c^*, although the calculations made were sufficient to have complete confidence in the existence of the singularity. By taking the positive sign of the square roots, as done in (12), we obtain *non-dual* saddle point flows, whereas by taking the negative sign we get the *dual* saddle point flows.

We may note that the singularity in the Falkner–Skan problem found numerically by Hartree [9] has recently [10] been demonstrated mathematically to exist. There is no doubt that the methods used in [10] could be used in the present problem to verify the singular behaviour as $c \to c^*$, and the latter's existence does suggest that for $c < c^*$ there are no solutions.

It should be noted that the dual solutions for $c > 1$ cannot be obtained from the dual solutions corresponding to $c < 1$: they are distinct solutions in their own right. However, we can still confine attention to the range $0 < c < 1$ by using the relations in (10), although the independent variable is $\eta = c^{\frac{1}{4}} Z$, i.e. the non-dimensionalisation of Grashof number and normal distance from the wall is based on b instead of a. As in [7], when (f_d, g_d, h_d) corresponding to $c > 1$ is transformed into $c < 1$ using equations (10), we denote the solution by $({}_*g, {}_*f, {}_*h)$.

Further examination of the numerical results suggests that there is a second singularity present as $c \to 0+$ along the transformed dual solutions. The singularity manifests itself by

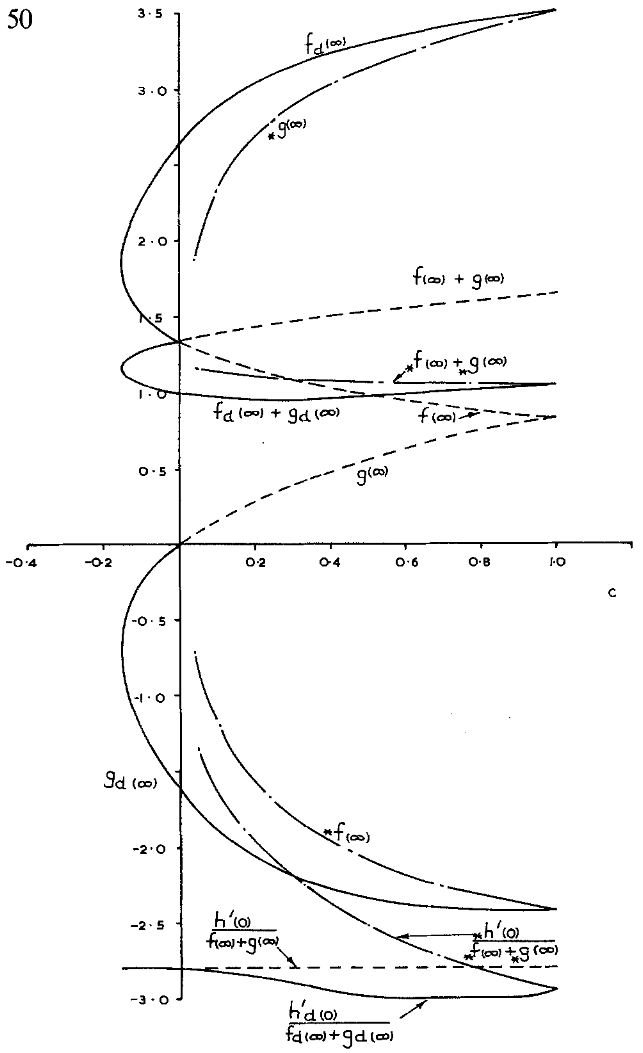


Figure 3. Flow details at the edge of the boundary layer. (See caption to figure 2 for notation etc.)

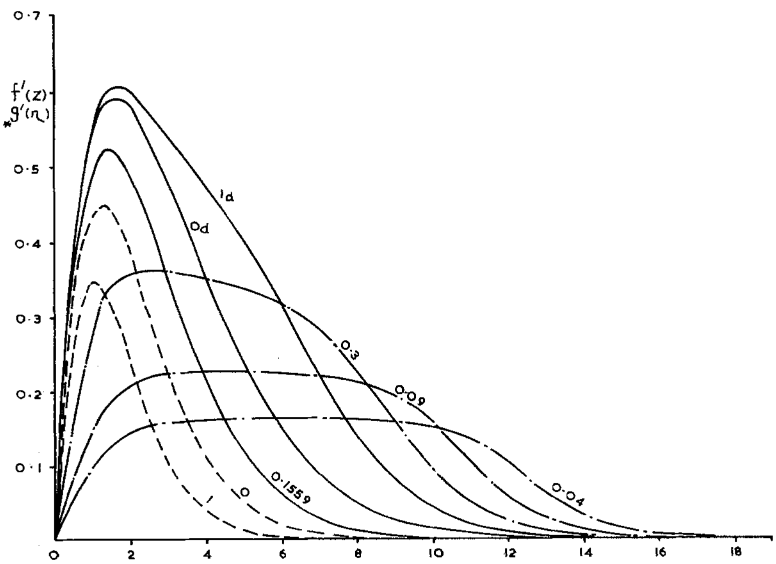


Figure 4. Velocity profiles $f'(Z)$ and $_*g'(\eta)$ for various values of c shown. (See caption to figure 2 for notation etc.)

way of $_*f''(0)$, $_*g''(0)$, $_*h'(0)$ behaving like $c^{\frac{1}{2}}$ and also by $_*f'(\eta)$, $_*g'(\eta)$, $_*h(\eta)$ decaying for η large progressively more slowly as $c \to 0+$. Because of the large ranges of integration that would clearly be necessary for very small values of c results are presented down to $c = 0.04$ only. This singularity as $c \to 0+$ is somewhat akin to that found by Stewartson [11] in the lower branch solutions of the Falkner–Skan equation as $\beta \to 0-$. We note he was able to show analytically that this bifurcation point did not give rise to yet further solutions but we have not proved this for the present problem.

It would appear that in all probability this second singularity will also exist in the forced convection problem considered in [7]; however the prime importance of the latter work was to demonstrate the existence of the dual solutions and it seems the authors did not integrate the relevant equations for sufficiently small c to appreciate the existence of the singularity. It does mean nevertheless that the dashed curve in figure 1 of [7] representing the variation of

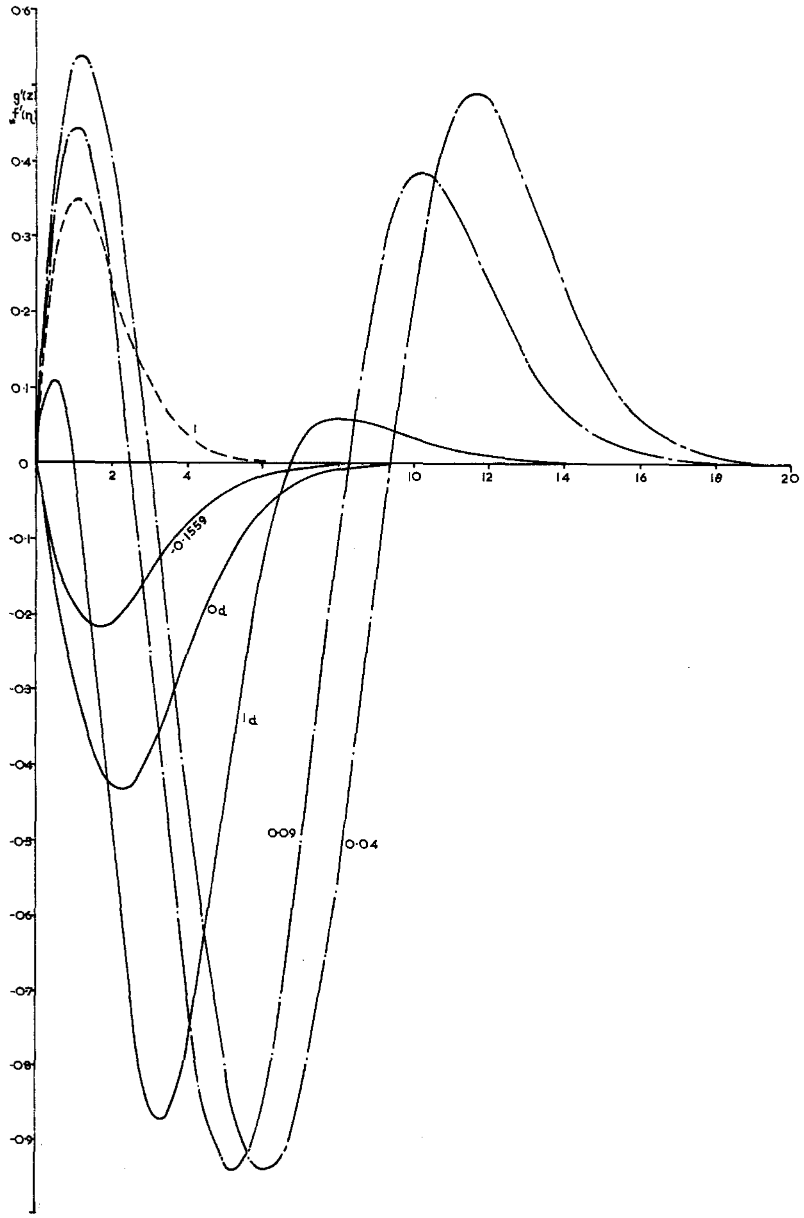


Figure 5. Velocity profiles $g'(Z)$ and $_*f'(\eta)$ for various values of c shown. (See caption to figure 2 for notation etc.)

$_*f''(0)$ with c does not describe the correct behaviour as $c \to 0+$.

Near $z=0$

$$w \sim -\nu a^3 G^{\frac{3}{4}}(f''(0)+g''(0))z^2$$

and since $f''(0)+g''(0)>0$ in all solutions presented, they are flows of *attachment* as expected. Further, since

$$J = \partial(\varepsilon_x, \varepsilon_y)/\partial(x, y) = \nu^2 a^6 G^{\frac{3}{2}} f''(0) g''(0)$$

and $f''(0)>0$ for all $c \geqq c^*$, it follows that J takes the same sign as $g''(0)$. Hence saddle point flows result from *non-dual* solutions when $0 > c \geqq c^*$, and from *dual* solutions when $c^* \leqq c < 0.38$. Nodal point flows result from dual solutions when $c > 0.38$ in addition to those catalogued by Poots.

Figures 4, 5 and 6 give the velocity profiles and the temperature variation for the various values of c indicated. Also shown for comparison are the axisymmetric solution ($c=1$) and the two-dimensional solution ($c=0$) in each case. Even a glance at these graphs reveals significant variations with c when compared with the variation of the non-dual nodal solutions: consider for example the non-dual variations of $f'(Z)$, $g'(Z)$, $h(Z)$ for the range $c^* \leqq c \leqq 0$.

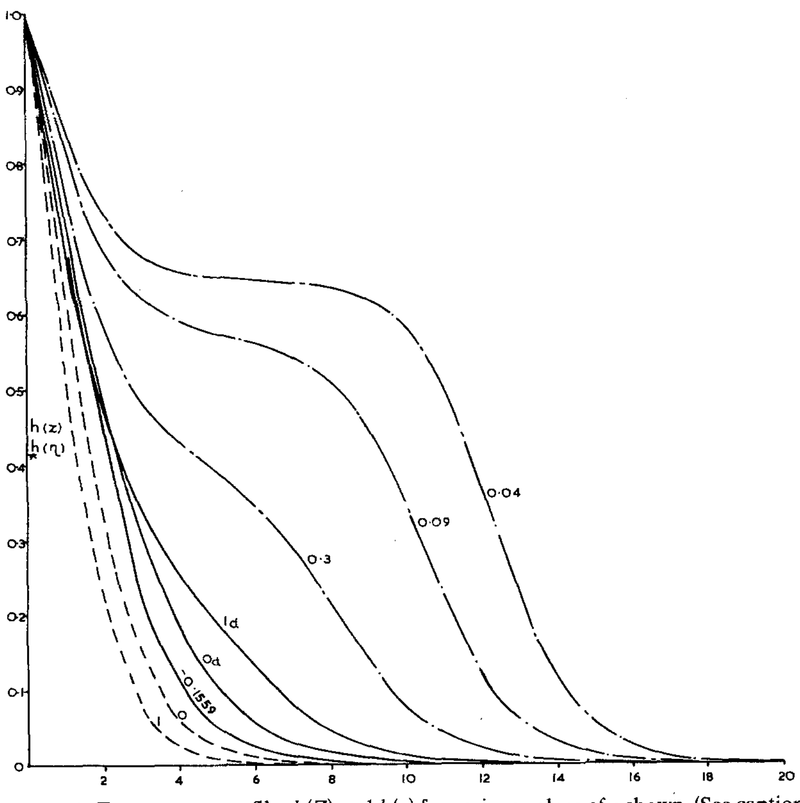


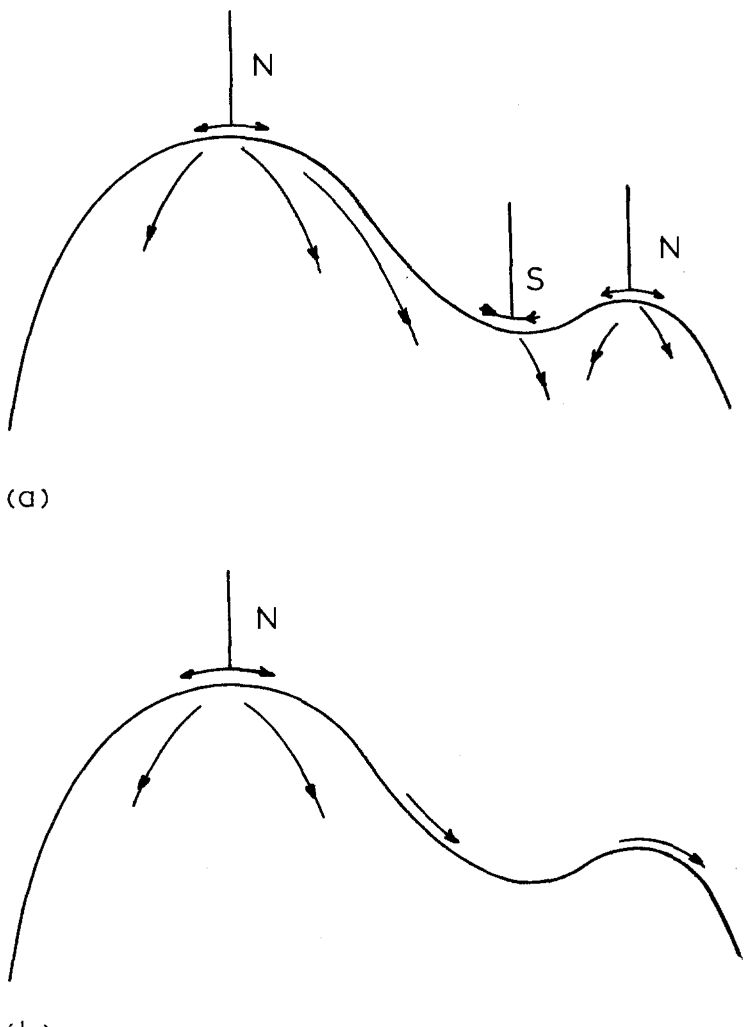
Figure 6. Temperature profiles $h(Z)$ and $_*h(\eta)$ for various values of c shown. (See caption to figure 2 for notation etc.)

4. Discussion

The existence of the singularity at $c=c^*$ is not without interest. It may reasonably be interpreted as the inability of the boundary layer to exist at values of $c<c^*$ for given σ, and that if indeed one proceeded with a three-dimensional boundary-layer calculation for free convectional flow past a body shaped as in figure 1, where the arrow now indicates the direction of gravity, $T_0 < T_\infty$ and $c<c^*$ at S, then one may expect to find that the boundary layer breaks down between N and S due to a singularity. Also the implication is that at the point of breakdown

both components of the skin friction and the heat transfer are non-zero. This type of behaviour has been suggested by Stewartson [12] in a not unrelated problem, although in the special case considered by Merkin [13] and Buckmaster [14] the only singularity found coincided with the position of zero skin-friction.

A problem that is relevant here concerns the two-dimensional free convectional flow about a heated circular cylinder. Switzer [15] has considered this situation in some detail but from the results presented no conclusion can be drawn about the existence of a singularity-presumably the assumption is that the fluid from either side of the cylinder collides at the upper generator and erupts into a plume. Indeed, one can prove (see Appendix) that, providing there is no singularity, then collision and consequent eruption *must* occur. However, we note that in [15], following Saville and Churchill [16], the meridional velocity component is written with an explicit dependence on the meridional co-ordinate in such a way that the boundary layer velocity vanishes at the lower *and* upper generator, i.e. an end boundary condition is imposed on a parabolic system of equations which is inconsistent. So although the method proposed in [16] is very suitable for some regions it would appear that some difficulty will be encountered near the upper generator.



(a)

(b)

Figure 7. Typical non-symmetric body which may give rise to a free convection flow (a) with two nodal points and one saddle point of attachment, or (b) with one nodal point of attachment only.

If one adopts the view that there is no breakdown of the boundary layer in such situations, and the appropriate boundary layer equations do not suggest otherwise, then for non-symmetrical bodies extreme care would be required to obtain physically meaningful results in any calculations. For example for symmetrical bodies like the circular cylinder or that illustrated in figure 1 (with $c < c^*$ at S⋆) the point of collision will be in the plane of symmetry, although for three-dimensional bodies like that illustrated in figure 7 there is the possibility of (i) a flow existing with a collision occurring (as in figure 7(a)) at a point to be determined, or (ii) a flow with only one nodal point of attachment as sketched in figure 7(b).

We may note here that in the forced problem of Davey [3] the assumption of three stagnation points for the body indicated in figure 1 leads to separation with a singularity present [4], when $c < -0.43$ at S. The suggestion made by Davey is that the outer flow in the region of S changes drastically as c changes from $-0.43+0$ to $-0.43-0$, so that the one saddle point flow at S is replaced by a *nodal* point flow at S which is flanked by two saddle point flows. This appears to be a very reasonable picture and is certainly not inconsistent with the conventional ideas on two-dimensional laminar boundary layer separation with its associated singularity—it is argued that the particular outer flow used is not correct and that it will be modified in such a way that the singularity is eliminated.

One of the difficulties in the free convection situation is that there is no outer flow to the appropriate order to which one could appeal for a suitable modification.

The sort of boundary layer collision that is envisaged above also occurs at the equator of a rotating sphere (Howarth [17]), although the flow in the eruption region has yet to be resolved even for this symmetrical situation [18]. We merely note that suction, injection, or modification of the geometry at one hemisphere will destroy the symmetry of the interaction (providing separation has not occurred) and the problem is presumably more difficult since even the point of collision will have to be determined.

We have no physical interpretation for the dual solutions. It is possible that certain of such solutions describe the non-symmetric collisions discussed above; although even if they do they may only be of theoretical interest in view of the ideas advanced in [10] where it is conjectured that a bifurcation point such as that at c^* is a margin of stability between stable and unstable regimes.

Finally, there is an interesting and immediate consequence of the singularity at c^* which can be interpreted in terms of a change in the Prandtl number. For, by the methods in [10], a singularity implies that if we denote the solution at $c = c^*$ by f^*, g^*, h^* it follows that the homogeneous equations

$$
\begin{aligned}
&F''' + (f^* + g^*)F'' + (F+G)f^{*''} - 2f^{*'}F' + H = 0\,,\\
&G''' + (f^* + g^*)G'' + (F+G)g^{*''} - 2g^{*'}G' + c^*H = 0\,,\\
&H''' + \sigma(F+G)h^{*'} + \sigma(f^* + g^*)H' = 0\,,
\end{aligned}
\tag{13}
$$

satisfying the homogeneous boundary conditions

$$
\begin{aligned}
&F(0) = F'(0) = G(0) = G'(0) = H(0) = 0\,,\\
&F'(Z) \to 0, \quad G'(Z) \to 0, \quad H(Z) \to 0 \text{ as } Z \to \infty
\end{aligned}
\tag{14}
$$

possess a non-trivial solution. Full details of the method are available in [10] and it suffices to remark that the "normalising" factor for this homogeneous solution is obtained at the next order terms. However a consequence of this is that by perturbing the solution about $c = c^*$, $\sigma = 0.72$ with respect to the Prandtl number, by writing $\sigma = 0.72 + \delta\sigma$, the existence of the solution to the above homogeneous problem implies that the variations of f, g, h *with* σ at c^* are of the form

$$
f = f^* + (\delta\sigma)^p \Phi + \ldots\,, \quad g = g^* + (\delta\sigma)^p \Gamma + \ldots\,, \quad h = h^* + (\delta\sigma)^p \Theta + \ldots\,,
\tag{15}
$$

⋆ It is assumed that if $c > c^*$ at S then the saddle point flows of §3 will be recovered.

where $0<p<1$ and Φ, Γ, Θ satisfy the same homogeneous problem as F, G, H respectively in (13) and (14). Different letters have been used for the perturbation terms because the "normalising" factor, obtained at a later stage of the expansion in (15), will be different. The sign of $\delta\sigma$ will also be obtained when the "normalising" factor is known. In all probability $p=\frac{1}{2}$ and so if the factor is real we must take $\delta\sigma>0$ whereas if it is totally complex we must take $\delta\sigma<0$: in either case the bifurcation point is moved but still exists.

For reasons that follow, it is suggested that the "normalising" factor in (15) is real so that c^* is a monotonic decreasing function of σ. The evidence for this concerns the behaviour at the two limiting values $\sigma\to\infty$ and $\sigma\to 0$. For $\sigma\gg 1$ the method of matched asymptotic expansions indicates that a viscous layer near the body exists such that

$$f(Z)\sim\chi^{-\frac{3}{4}}f_0(\xi)+\dots,$$
$$g(Z)\sim\chi^{-\frac{3}{4}}f_0(\xi)+\dots,$$
$$h(Z)\sim h_0(\xi)+\dots,$$

where $\xi=\chi^{\frac{1}{4}}Z$, $\chi=\sigma(1+c)$ and f_0, h_0 are known from the two-dimensional problem (details are given in [8]), and it is conjectured from this result that for $\sigma\gg 1$, the value of c, beyond which no solutions exist, tends to -1. Further, for $0<\sigma\ll 1$, a similar analysis leads to an inviscid layer such that

$$f\sim\sigma^{-\frac{1}{2}}F_0(\eta)+\dots,$$
$$g\sim\sigma^{-\frac{1}{2}}G_0(\eta)+\dots,$$
$$h\sim H_0(\eta)+\dots,$$

where $\eta=\sigma^{\frac{1}{2}}Z$ and F_0, G_0, H_0, satisfy

$$(F_0+G_0)F_0''-F_0'^2+H_0=0\,,$$
$$(F_0+G_0)G_0''-G_0'^2+cH_0=0\,,$$
$$H_0''+(F_0+G_0)H_0'=0\,,$$

subject to

$$F_0(0)=G_0(0)=0\,,\quad H_0(0)=1\,,$$
$$F_0'(\eta)\to 0\,,\quad G_0'(\eta)\to 0\,,\quad H_0(\eta)\to 0 \text{ as } \eta\to\infty\,.$$

However, a consequence is that

$$F_0'(0)=1\,,\quad G_0'(0)=c^{\frac{1}{2}}\,,$$

which again leads to the obvious conjecture that as $\sigma\to 0$ there are no solutions for $c<0$.

Appendix

The purpose here is to show that the boundary layers that start at the lower generator and move up along both sides of a heated horizontal circular cylinder do collide at the upper generator, *provided* there is no singularity in the boundary layer equations. The proof offered here follows the method suggested by Howarth [17] where the boundary layers of a rotating sphere are shown to collide.

The boundary layer equations for free convectional flow over a circular cylinder are (see [15] for example)

$$u\frac{\partial u}{\partial\theta}+v\frac{\partial u}{\partial y}=\Theta\sin\theta+\frac{\partial^2 u}{\partial y^2}\tag{A.1}$$

$$u \frac{\partial \Theta}{\partial \theta} + v \frac{\partial \Theta}{\partial y} = \sigma^{-1} \frac{\partial^2 \Theta}{\partial y^2} \tag{A.2}$$

where θ is the angle subtended at the axis of the cylinder and measured from the lower generator, y is the distance measured normal to the surface of the cylinder, (u, v) are velocities in the directions (θ, y) increasing and Θ is a measure of the temperature. The boundary conditions for an isothermal surface are

$$\begin{aligned} &u = v = 0\,, \quad \Theta = 1 \text{ at } y = 0\,, \\ &u \to 0\,, \quad \Theta \to 0 \text{ as } y \to \infty\,. \end{aligned} \tag{A.3}$$

However, if at the edge of the boundary layer we denote the normal velocity by $v_\infty = v_\infty(\theta)$ then (A.1) and (A.2) imply in view of (A.3) that

$$\frac{\partial^2 u}{\partial y^2} - v_\infty \frac{\partial u}{\partial y} = -\Theta \sin \theta\,,$$

$$\frac{\partial^2 \Theta}{\partial y^2} - \sigma v_\infty \frac{\partial \Theta}{\partial y} = 0\,.$$

It then follows that

$$\Theta = A(\theta) e^{\sigma y v_\infty}\,,$$

$$u = \frac{A(\theta) \cdot \sin \theta}{\sigma v_\infty^2 (1-\sigma)} e^{\sigma y v_\infty} + B(\theta) e^{y v_\infty}\,,$$

for large y and $\sigma \neq 1$. Hence in order to satisfy the boundary conditions in (A.3) we require $v_\infty < 0$, i.e. in-flow. It therefore follows that the boundary layers collide.

REFERENCES

[1] L. Rosenhead (Ed.), *Laminar Boundary Layers*, Oxford University Press (1963).

[2] L. Howarth, The Boundary layer in Three Dimensional Flow—Part II. The Flow near a Stagnation Point, *Phil. Mag. Ser.*, 7, 42 (1951) 1433–1440.

[3] A. Davey, Boundary-layer flow at a saddle point of attachment. *J. Fluid Mech.*, 4 (1961) 593–610.

[4] W. H. H. Banks, A three-dimensional laminar boundary layer calculation, *J. Fluid Mech.*, 28 (1967) 769–792.

[5] J. C. Cooke and A. Robins, Boundary-layer flow between nodal and saddle points of attachment, *J. Fluid Mech.*, 41 (1970) 823–835.

[6] G. Poots, Laminar free convection near the lower stagnation point on an isothermal curved surface. *Int. J. Heat Mass Transfer*, 7 (1964) 863–874.

[7] D. Schofield and A. Davey, Dual solutions of the boundary-layer equations at a point of attachment, *J. Fluid Mech.*, 30 (1967) 809–811.

[8] W. H. H. Banks, Three-dimensional Free convection Near a Two-Dimensional Isothermal Surface, *J. Engng. Math.*, 6 (1972) 109–115.

[9] D. R. Hartree, On an equation occurring in Falkner and Skan's approximate treatment of the equations of the boundary layer, *Proc. Camb. Phil. Soc.*, 33 (1937) 223–239.

[10] W. H. H. Banks and P. G. Drazin, Perturbation methods in boundary-layer theory. *J. Fluid Mech.*, 58 (1973) 763–775.

[11] K. Stewartson, Further solutions of the Falkner–Skan equation. *Proc. Camb. Phil. Soc.*, 50 (1954) 454–465.

[12] K. Stewartson, The behaviour of a laminar compressible boundary layer near a point of zero skin friction, *J. Fluid Mech.*, 12 (1962) 117–128.

[13] J. H. Merkin, The effect of buoyancy forces on the boundary layer flow over a semi-infinite vertical flat plate in a uniform free stream, *J. Fluid Mech.*, 35 (1969) 439–450.

[14] J. Buckmaster, The behaviour of a laminar compressible boundary layer on a cold wall near a point of zero skin friction, *J. Fluid Mech.*, 44 (1970) 237–247.

[15] K. P. Switzer, The effect of buoyancy on the boundary layer about a heated horizontal circular cylinder in axial streaming, *J. Fluid Mech.*, 44 (1970) 529–543.

[16] D. A. Saville and S. W. Churchill, Laminar free convection in boundary layers near horizontal cylinders and vertical axisymmetric bodies, *J. Fluid Mech.*, 29 (1967) 391–399.

[17] L. Howarth, Note on the boundary layer on a rotating sphere, *Phil. Mag. Ser.*, 7, 42 (1951) 1308–1315.

[18] W. H. H. Banks, The boundary layer on a rotating sphere, *Quart. J. Mech. Appl. Math.*, 18 (1965) 443–454.